Determinant Representation of Correlation Functions for the $U_q(gl(1|1))$ Free Fermion Model

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Abstract

With the help of the factorizing F-matrix, the scalar products of the $U_q(gl(1|1))$ free fermion model are represented by determinants. By means of these results, we obtain the determinant representations of correlation functions of the model.

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I Introduction

The computation of correlation functions is one of the challenging problem in the theory of quantum integrable lattice models [1, 2]. In this paper, we compute the correlation functions of the free fermion model by means of the algebraic Bethe ansatz method [1, 2, 3]. Our computation are based on the recent progress on the Drinfeld twists. Working in the F-bases provided by the F-matrices (Drinfeld twists), the authors in [4, 5] managed to derive the determinant representations of the form factors and correlation functions of the XXX and XXZ models in the framework of algebraic Bethe ansatz.

Recently we have constructed the Drinfeld twists for both the rational $gl(m|n)$ and the quantum $U_q(gl(m|n))$ supersymmetric models and resolved the hierarchy of their nested Bethe vectors in the F-basis [6, 7, 8]. These results serve as the basis of our computation in this paper of the correlation functions of the $U_q(gl(1|1))$ free fermion model.

Correlation functions of the free fermion model based on the XX0 spin chain (XY model [9]) with periodic boundary condition were studied in [10]-[15]. As is seen in section VI, by using the Jordan-Wigner transform, our $U_q(gl(1|1))$ free fermion model is equivalent to a twisted XX0 model, and the one-point functions we obtained (see (V.5) and (V.7) below) give the m-point correlation functions of the twisted XX0 model (see e.g. (VI.6)).

The present paper is organized as follows. In section II, we review the background of the $U_q(gl(1|1))$ model and its algebraic Bethe ansatz. In section III, we construct the Drinfeld twists for the model. In section IV, we obtain the determinant representation of the scalar products of the $U_q(gl(1|1))$ Bethe states. Then in section V, we compute correlation functions of the local fermionic operators of the model. We conclude the paper by offering some discussions in section VI.

II $U_q(gl(1|1))$ free fermion model

II.1 The background of the model

Let V be the 2-dimensional $U_q(gl(1|1))$-module and $R \in End(V \otimes V)$ the R-matrix associated with this module. V is Z_2-graded, and in the following we choose the FB grading for V, i.e. $[1] = 1, [2] = 0$. The R-matrix depends on the difference of two spectral parameters u_1 and

u_2 associated with the two copies of V, and is, in the FB grading, given by

$$R_{12}(u_1,u_2)=R_{12}(u_1-u_2) \quad = \quad \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ 0 & a_{12} & b^{+}_{12} & 0 \\ 0 & b^{-}_{12} & a_{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{II.1}$$

where

$$a_{12}=a(u_1,u_2)\equiv\frac{\sinh(u_1-u_2)}{\sinh(u_1-u_2+\eta)}, \qquad b^{\pm}_{12}=b^{\pm}(u_1,u_2)\equiv\frac{e^{\pm(u_1-u_2)}\sinh\eta}{\sinh(u_1-u_2+\eta)},$$
$$c_{12}=c(u_1,u_2)\equiv\frac{\sinh(u_1-u_2-\eta)}{\sinh(u_1-u_2+\eta)} \tag{II.2}$$

with $\eta\in C$ being the crossing parameter. One can easily check that the R-matrix satisfies the unitary relation

$$R_{21}R_{12}=1. \tag{II.3}$$

Here and throughout $R_{ij}\equiv R_{ij}(u_i,u_j)$. The R-matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{II.4}$$

In terms of the matrix elements defined by

$$R(u)(v^{i'}\otimes v^{j'})=\sum_{i,j}R(u)^{i'j'}_{ij}(v^{i}\otimes v^{j}), \tag{II.5}$$

the GYBE reads

$$\begin{aligned}&\sum_{i',j',k'}R(u_1-u_2)^{i'j'}_{ij}R(u_1-u_3)^{i''k'}_{i'k}R(u_2-u_3)^{j''k''}_{j'k'}(-1)^{[j']([i']+[i''])}\\ =\;&\sum_{i',j',k'}R(u_2-u_3)^{j'k'}_{jk}R(u_1-u_3)^{i'k''}_{ik'}R(u_1-u_2)^{i''j''}_{i'j'}(-1)^{[j']([i]+[i'])}.\end{aligned} \tag{II.6}$$

The quantum monodromy matrix $T(u)$ of the free fermion model on a lattice of length N is defined as

$$T_0(u)=R_{0N}(u,z_N)R_{0N-1}(u,z_{N-1})...R_{01}(u,z_1), \tag{II.7}$$

where the index 0 refers to the auxiliary space and $\{z_i\}$ are arbitrary inhomogeneous parameters depending on site i. $T(u)$ can be represented in the auxiliary space as the 2×2 matrix whose elements are operators acting on the quantum space $V^{\otimes N}$:

$$T_0(u)=\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u)\end{pmatrix}_{(0)}. \tag{II.8}$$

By using the GYBE, one may prove that the monodromy matrix satisfies the GYBE

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \tag{II.9}$$

or in matrix form,

$$\begin{aligned}&\sum_{i',j'} R(u-v)_{ij}^{i'j'} T(u)_{i'}^{i''} T(v)_{j'}^{j''} (-1)^{[j']([i']+[i''])} \\ &\qquad = \sum_{i',j'} T(v)_j^{j'} T(u)_i^{i'} R(u-v)_{i'j'}^{i''j''} (-1)^{[j']([i]+[i'])}.\end{aligned} \tag{II.10}$$

Define the transfer matrix $t(u)$

$$t(u) = str_0 T_0(u), \tag{II.11}$$

where str_0 denotes the supertrace over the auxiliary space. With the help of the GYBE, one may check that the transfer matrix satisfies the commutation relation $[t(u), t(v)] = 0$, ensuring the integrability of the system. The transfer matrix gives the Hamiltonian of the system:

$$\begin{aligned} H &= \frac{d\ln t(u)}{du}|_{u=0} \\ &= \frac{1}{\sinh\eta}\sum_{j=1}^{N}\Big(E_{(j)}^{12}E_{(j+1)}^{21} + E_{(j)}^{21}E_{(j+1)}^{12} - 2\cosh\eta E_{(j)}^{11}E_{(j+1)}^{11} \\ &\qquad -(e^{\eta}E_{(j)}^{11}E_{(j+1)}^{22} + e^{-\eta}E_{(j)}^{22}E_{(j+1)}^{11})\Big), \end{aligned} \tag{II.12}$$

where $E_{(k)}^{ij}$ are generators, which act on the kth space, of the superalgebra $U_q(gl(1|1))$.

Using the standard fermionic representation

$$E_{(k)}^{12} = c_k, \quad E_{(k)}^{21} = c_k^\dagger, \quad E_{(k)}^{11} = 1 - n_k, \quad E_{(k)}^{22} = n_k, \quad n_k = c_k^\dagger c_k, \tag{II.13}$$

the Hamiltonian can be rewritten as

$$H = \frac{1}{\sinh\eta}\sum_{j=1}^{N}\left(c_j c_{j+1}^\dagger + c_j^\dagger c_{j+1} - 2\cosh\eta(1-n_j)\right). \tag{II.14}$$

II.2 Algebraic Bethe ansatz

The transfer matrix (II.11) can be diagonalized by using the algebra Bethe ansatz. Define the Bethe state of the system

$$\Phi_N(v_1, v_2, \ldots, v_n) = \prod_{i=1}^{n} C(v_i)|0>, \tag{II.15}$$

where $|0>$ is the pseudo-vacuum,

$$|0>=\prod_{k=1}^{N}\begin{pmatrix}0\\1\end{pmatrix}_{(k)} \tag{II.16}$$

and the index (k) indicates the kth space.

Applying the elements of the monodromy matrix (II.8) to the pseudo-vacuum $|0>$ and its dual, we easily obtain

$$\begin{aligned}&B(u)|0>=0, \quad <0|C(u)=0, \quad D|0>=|0>, \quad <0|D(u)=<0|,\\ &A(u)|0>=\prod_{i=1}^{N}a(u,z_i)|0>, \quad <0|A(u)=\prod_{i=1}^{N}a(u,z_i)<0|.\end{aligned} \tag{II.17}$$

With the help of the GYBE (II.9), we obtain the commutation relations between the elements of the monodromy matrix

$$C(u)C(v) = -c(u,v)C(v)C(u), \tag{II.18}$$

$$D(u)D(v) = D(v)D(u), \tag{II.19}$$

$$A(u)C(v) = \frac{c(u,v)}{a(u,v)}C(v)A(u)+\frac{b^{+}(u,v)}{a(u,v)}C(u)A(v), \tag{II.20}$$

$$D(u)C(v) = \frac{1}{a(v,u)}C(v)D(u)-\frac{b^{-}(v,u)}{a(v,u)}C(u)D(v), \tag{II.21}$$

$$\begin{aligned}B(u)C(v) &= -C(v)B(u)+\frac{b^{+}(u,v)}{a(u,v)}[D(v)A(u)-D(u)A(v)]\\ &= -C(v)B(u)+\frac{b^{+}(u,v)}{a(u,v)}[D(u)t(v)-D(v)t(u)].\end{aligned} \tag{II.22}$$

Thus applying the transfer matrix $t(u)=D(u)-A(u)$ to the Bethe state and using the commutation relations repeatedly, we obtain the eigenvalues of $t(u)$ as

$$t(u)\Phi_N=\Lambda(u,\{v_k\})\Phi_N=\left[\prod_{k=1}^{n}\frac{1}{a(v_k,u)}-\prod_{j=1}^{N}a(u,z_j)\prod_{k=1}^{n}\frac{c(u,v_k)}{a(u,v_k)}\right]\Phi_N \tag{II.23}$$

providing $v_k\,(k=1,2,\ldots,n)$ satisfying the Bethe ansatz equations (BAE)

$$\prod_{j=1}^{N}a(v_k,z_j)=1. \tag{II.24}$$

For late use, we define the state of the free fermion chain of length N

$$|a_1a_2\ldots a_N>=|a_1>_{(1)}|a_2>_{(2)}\ldots|a_N>_{(N)} \tag{II.25}$$

and its dual

$$|a_1a_2\ldots a_N>^{\dagger}=<a_N|_{(N)}<a_{N-1}|_{(N-1)}\ldots<a_1|_{(1)}\equiv<a_Na_{N-1}\ldots a_1|. \tag{II.26}$$

III Drinfeld twists of the model

III.1 Factorizing F-matrix and its inverse

Following [4], we now introduce the notation $R^{\sigma}_{1\ldots N}$, where σ is any element of the permutation group $\mathcal{S}_N$. We note that we may rewrite the GYBE as

$$R^{\sigma_{23}}_{23} T_{0,23} = T_{0,32} R^{\sigma_{23}}_{23}, \tag{III.1}$$

where $T_{0,23} \equiv R_{03}R_{02}$ and σ_{23} is the transposition of space labels (2,3). It follows that $R^{\sigma}_{1\ldots N}$ is a product of elementary R-matrices [4, 6], corresponding to a decomposition of σ into elementary transpositions. With the help of the GYBE, one may generalize (III.1) to a N-fold tensor product of spaces

$$R^{\sigma}_{1\ldots N} T_{0,1\ldots N} = T_{0,\sigma(1\ldots N)} R^{\sigma}_{1\ldots N}, \tag{III.2}$$

where $T_{0,1\ldots N} \equiv R_{0N} \ldots R_{01}$. This implies the "decomposition" law

$$R^{\sigma'\sigma}_{1\ldots N} = R^{\sigma}_{\sigma'(1\ldots N)} R^{\sigma'}_{1\ldots N}, \tag{III.3}$$

for a product of two elements in $\mathcal{S}_N$. Note that $R^{\sigma}_{\sigma'(1\ldots N)}$ satisfies the relation

$$R^{\sigma}_{\sigma'(1\ldots N)} T_{0,\sigma'(1\ldots N)} = T_{0,\sigma'\sigma(1\ldots N)} R^{\sigma}_{\sigma'(1\ldots N)}. \tag{III.4}$$

Let us write the elements of $R^{\sigma}_{1\ldots N}$ as

$$(R^{\sigma}_{1\ldots N})^{\alpha_{\sigma(N)}\cdots\alpha_{\sigma(1)}}_{\beta_N\ldots\beta_1}, \tag{III.5}$$

where the labels in the upper indices are permuted relative to the lower indices according to σ.

We proved in [6, 7, 8] that for the R-matrix $R^{\sigma}_{1\ldots N}$, there exists a non-degenerate lower-diagonal F-matrix (the Drinfeld twist) satisfying the relation

$$F_{\sigma(1\ldots N)}(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) R^{\sigma}_{1\ldots N}(z_1, \ldots, z_N) = F_{1\ldots N}(z_1, \ldots, z_N). \tag{III.6}$$

Explicitly,

$$F_{1,\ldots N} = \sum_{\sigma\in\mathcal{S}_N} \sum^{*}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(N)}} \prod_{j=1}^{N} P^{\alpha_{\sigma(j)}}_{\sigma(j)} S(c,\sigma,\alpha_\sigma) R^{\sigma}_{1\ldots N}, \tag{III.7}$$

where the sum $\sum^*$ is over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$:

$$\begin{aligned} \alpha_{\sigma(i+1)} &\geq \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) > \sigma(i), \\ \alpha_{\sigma(i+1)} &> \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) < \sigma(i) \end{aligned} \tag{III.8}$$

and the c-number function $S(c, \sigma, \alpha_\sigma)$ is given by

$$S(c, \sigma, \alpha_\sigma) \equiv \exp\left\{ \frac{1}{2} \sum_{l>k=1}^{N} \left(1 - (-1)^{[\alpha_{\sigma(k)}]}\right) \delta_{\alpha_{\sigma(k)}, \alpha_{\sigma(l)}} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\}. \tag{III.9}$$

The inverse of the F-matrix is given by

$$F^{-1}_{1\ldots N} = F^*_{1\ldots N} \prod_{i<j} \Delta^{-1}_{ij} \tag{III.10}$$

with

$$\Delta_{ij} = \text{diag}\left((1 + c_{ij})(1 + c_{ji}), a_{ji}, a_{ij}, 1\right) \tag{III.11}$$

and

$$F^*_{1\ldots N} = \sum_{\sigma \in \mathcal{S}_N} \sum^{**}_{\alpha_{\sigma(1)} \cdots \alpha_{\sigma(N)}} S(c, \sigma, \alpha_\sigma) R^{\sigma^{-1}}_{\sigma(1\ldots N)} \prod_{j=1}^{N} P^{\alpha_{\sigma(j)}}_{\sigma(j)}, \tag{III.12}$$

where the sum $\sum^{**}$ is taken over all possible α_i which satisfies the following non-increasing constraints:

$$\begin{aligned} \alpha_{\sigma(i+1)} &\leq \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) < \sigma(i), \\ \alpha_{\sigma(i+1)} &< \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) > \sigma(i). \end{aligned} \tag{III.13}$$

III.2 Symmetric representation of the Bethe state

The non-degeneracy of the F-matrix means that its column vectors form a complete basis, which is called the F-basis. By the procedure in [8], we find that in the F-basis, the simple generators of the superalgebra $U_q(gl(1|1))$ have the symmetric form:

$$\begin{aligned} \tilde{E}^{12} &= F_{12\ldots N} E^{12} F^{-1}_{12\ldots N} \\ &= \sum_{i=1}^{N} E^{12}_{(i)} \otimes_{j \neq i} \text{diag}\left(2e^{-\eta} \cosh \eta, e^{-\eta}\right)_{(j)}, \end{aligned} \tag{III.14}$$

$$\begin{aligned} \tilde{E}^{21} &= F_{12\ldots N} E^{21} F^{-1}_{12\ldots N} \\ &= \sum_{i=1}^{N} E^{12}_{(i)} \otimes_{j \neq i} \text{diag}\left(e^{\eta}(2a_{ji} \cosh \eta)^{-1}, e^{\eta} a^{-1}_{ji}\right)_{(j)}. \end{aligned} \tag{III.15}$$

Similarly, the diagonal element $D(u)$ of the monodromy matrix in the F-basis is given by

$$\tilde{D}(u) \ = \ F_{12\ldots N}D(u)F^{-1}_{12\ldots N} = \otimes_{j=1}^{N}\text{diag}\,(a_{oj},1)\,, \qquad \text{(III.16)}$$

where $a_{0j} \equiv a(u,z_j)$.

Then, the creation and annihilation operators $C(u)$ and $B(u)$ read, in the F-basis,

$$\begin{aligned}\tilde{C}(u) &= F_{12\ldots N}C(u)F^{-1}_{12\ldots N} = (q^{-1}\tilde{E}^{12}_{(i)}\tilde{D}(u) - \tilde{D}(u)\tilde{E}^{12}_{(i)})q^{-\sum_{i=1}^{N}h_{(i)}} \\ &= \sum_{i=1}^{N} b^-_{0i}E^{12}_{(i)} \otimes_{j\neq i} \text{diag}\,(2a_{0j}\cosh\eta, 1)_{(j)}\,, \qquad \text{(III.17)}\\ \tilde{B}(u) &= F_{12\ldots N}B(u)F^{-1}_{12\ldots N} = q^{\sum_{i=1}^{N}h_{(i)}}(\tilde{E}^{21}\tilde{D} - q\tilde{D}\tilde{E}^{21}) \\ &= -\sum_{i=1}^{N} b^+_{0i}E^{21}_{(i)} \otimes_{j\neq i} \text{diag}\left(a_{0j}(2a_{ji}\cosh\eta)^{-1}, a^{-1}_{ji}\right)_{(j)}\,, \qquad \text{(III.18)}\end{aligned}$$

where $b^{\pm}_{0j} \equiv b^{\pm}(u,z_j)$, $q = e^{\eta}$ and $h \equiv -E^{11} - E^{22}$.

Acting the F-matrix (III.7) on the state (II.16), one sees that the pseudo-vacuum is invariant. Therefore in the F-basis, the Bethe state (II.15) becomes,

$$\tilde{\Phi}_N(v_1,v_2,\ldots,v_n) \equiv F_{1\ldots N}\Phi_N(v_1,\ldots,v_n) = \prod_{i=1}^{n}\tilde{C}(v_n)|0> . \qquad \text{(III.19)}$$

Substituting (III.17) into (III.19), we obtain

$$\tilde{\Phi}_N(v_1,\ldots,v_n) = (2\cosh\eta)^{\frac{n(n-1)}{2}} \sum_{i_1<\ldots<i_n} B^-_n(v_1,\ldots,v_n|z_{i_1},\ldots,z_{i_n})E^{12}_{(i_1)}\ldots E^{12}_{(i_n)}\,|0> , \qquad \text{(III.20)}$$

where

$$\begin{aligned}B^{\pm}_n(v_1,\ldots,v_n|z_{i_1},\ldots,z_{i_n}) &= \sum_{\sigma\in\mathcal{S}_n}\text{sign}(\sigma)\prod_{k=1}^{n} b^{\pm}(v_k,z_{i_{\sigma(k)}})\prod_{l=k+1}^{n} a(v_k,z_{i_{\sigma(l)}}) \\ &= \det\mathcal{B}^{\pm}(\{v_k\},\{z_j\}) \qquad \text{(III.21)}\end{aligned}$$

with $\mathcal{B}^{\pm}(\{v_i\},\{z_j\})$ being a $n\times n$ matrix with matrix elements

$$\mathcal{B}^{\pm}_{\alpha\beta} = b^{\pm}(v_\alpha,z_\beta)\prod_{\gamma=1}^{\alpha-1} a(v_\gamma,z_\beta). \qquad \text{(III.22)}$$

Similarly, acting $\tilde{B}(u_n)\ldots\tilde{B}(u_1)$ on the dual pseudo-vacuum state, we have,

$$\begin{aligned}< 0|\tilde{B}(u_n)\ldots\tilde{B}(u_1) &= (-1)^n(2\cosh\eta)^{\frac{-n(n-1)}{2}}\sum_{i_1<\ldots<i_n}\prod_{l=1}^{n}\prod_{k=1,\neq i_l}^{N} a^{-1}(z_k,z_{i_l}) \\ &\quad \times\det\mathcal{B}^{+}(\{v_k\},\{z_{i_j}\}) < 0|E^{21}_{(i_n)}\ldots E^{21}_{(i_1)}\ . \qquad \text{(III.23)}\end{aligned}$$

IV Determinant representation of the scalar product of the Bethe states

In [2, 5], the authors gave the determinant representation of the scalar product of the Bethe state for the spin-1/2 XXZ model. In this section, we derive the determinant representation of the scalar product of the $U_q(gl(1|1))$ Bethe states defined by

$$S_n(\{u_j\},\{v_k\}) =< 0|B(u_n)\ldots B(u_1)C(v_1)\ldots C(v_n)|0> . \tag{IV.1}$$

The F-invariance of the pseudo-vacuum state $|0>$ and its dual state $<0|$ leads to

$$S_n(\{u_j\},\{v_k\}) =< 0|\tilde{B}(u_n)\ldots \tilde{B}(u_1)\tilde{C}(v_1)\ldots \tilde{C}(v_n)|0> . \tag{IV.2}$$

Following [5], we define

$$\begin{aligned}&G^{(m)}(\{v_k\},u_1,\ldots,u_m,i_{m+1},\ldots,i_n)\\ &=< i_n,\ldots,i_{m+1}|\tilde{B}(u_m)\ldots\tilde{B}(u_1)\tilde{C}(v_1)\ldots\tilde{C}(v_n)|0>,\end{aligned} \tag{IV.3}$$

where i_k $(k=m+1,\ldots,n)$, ordered as $i_{m+1}<\ldots<i_n$, indicate the positions having state $\begin{pmatrix}1\\0\end{pmatrix}$, and other positions have state $\begin{pmatrix}0\\1\end{pmatrix}$. One sees that when $m=n$, $G^{(n)}=S_n$. Inserting a complete set and noticing (III.18), (IV.3) becomes

$$\begin{aligned}&G^{(m)}(\{v_k\},u_1,\ldots,u_m,i_{m+1},\ldots,i_n)\\ &=\sum_{j\neq i_{m+1},\ldots,i_n}^{N} < i_n,\ldots,i_{m+1}|\tilde{B}(u_m)|i_{m+1},\ldots,i_{m+p},j,i_{m+p+1},\ldots,i_n>\\ &\quad\times G^{(m-1)}(\{v_k\},u_1,\ldots,u_{m-1},i_{m+1},\ldots,i_{m+p},j,i_{m+p+1},\ldots,i_n).\end{aligned} \tag{IV.4}$$

In view of (III.18), we have

$$\begin{aligned}&< i_n,\ldots,i_{m+1}|\tilde{B}(u_m)|i_{m+1},\ldots,i_{m+p},j,i_{m+p+1},\ldots,i_n>\\ &=-(2\cosh\eta)^{-(n-m)}\cdot(-1)^p\, b^+(u_m,z_j)\prod_{k\neq j}^{N}a^{-1}(z_k,z_j)\prod_{l=m+1}^{n}a(u_m,z_{i_l}).\end{aligned} \tag{IV.5}$$

With the help of (III.20), we obtain $G^{(0)}$:

$$\begin{aligned}G^{(0)}(\{v_k\},i_1,\ldots,i_n) &=< i_n,\ldots,i_1|\prod_{k=1}^{n}\tilde{C}(v_k)|0>\\ &= (2\cosh\eta)^{\frac{n(n-1)}{2}}\det\mathcal{B}^-(\{v_k\},\{z_{i_l}\}).\end{aligned} \tag{IV.6}$$

We now compute $G^{(1)}$ by using the recursion relation (IV.4). Substituting (IV.5) and (IV.6) into (IV.4), we obtain

$$
\begin{aligned}
& G^{(1)}(\{v_k\}, u_1, i_2, \ldots, i_n) \\
&= \sum_{j \neq i_2,\ldots,i_n}^{N} < i_n, \ldots, i_2|\tilde{B}(u_1)|i_2, \ldots, i_{p+1}, j, i_{p+2}, \ldots, i_n > \\
&\quad \times G^{(0)}(\{v_k\}, i_2, \ldots, i_{p+1}, j, i_{p+2}, \ldots, i_n) \\
&= -(2\cosh\eta)^{\frac{(n-1)(n-2)}{2}} \sum_{j \neq i_2,\ldots,i_n}^{N} (-1)^p \, b^+(u_1, z_j) \prod_{k \neq j}^{N} a^{-1}(z_k, z_j) \prod_{l=2}^{n} a(u_1, z_{i_l}) \\
&\quad \times \det\mathcal{B}^-(\{v_k\}, z_{i_2}, \ldots, z_{i_{p+1}}, z_j, z_{i_{p+2}}, \ldots, z_{i_n}), \quad (k = 1, \cdots, n). \qquad \text{(IV.7)}
\end{aligned}
$$

Let $v_k\,(k = 1, \ldots, n)$ label the row and $z_l\,(l = i_2, \ldots, j, \ldots, i_n)$ label the column of the matrix $\mathcal{B}^-$. From (IV.6), one sees that the column indices in (IV.7) satisfy the sequence $i_2 < \ldots < j < \ldots < i_n$. Therefore, moving the column j in the matrix $\mathcal{B}^-$ to the first column, we have

$$
\begin{aligned}
& G^{(1)}(\{v_k\}, u_1, i_2, \ldots, i_n) \\
&= -(2\cosh\eta)^{\frac{(n-1)(n-2)}{2}} \sum_{j \neq i_1,\ldots,i_n}^{N} b^+(u_1, z_j) \prod_{k \neq j}^{N} a^{-1}(z_k, z_j) \prod_{l=2}^{n} a(u_1, z_{i_l}) \\
&\qquad \times \det\mathcal{B}(\{v_k\}, z_j, z_{i_2}, \ldots, z_{i_n}) \\
&= -(2\cosh\eta)^{\frac{(n-1)(n-2)}{2}} \det(\mathcal{B}^-)^{(1)}(\{v_k\}, u_1, z_{i_2}, \ldots, z_{i_n}), \qquad \text{(IV.8)}
\end{aligned}
$$

where the matrix $(\mathcal{B}^-)^{(1)}(\{v_k\}, u_1, z_{i_2}, \ldots, z_{i_n})$ is given by

$$
\begin{aligned}
(\mathcal{B}^-_{\alpha\beta})^{(1)} &= a(u_1, z_{i_\beta})\mathcal{B}^-_{\alpha\beta} \qquad \text{for } \beta \geq 2, \qquad \text{(IV.9)} \\
(\mathcal{B}^-_{\alpha 1})^{(1)} &= \sum_{j \neq i_2,\ldots,i_n}^{N} b^+(u_1, z_j) b^-(v_\alpha, z_j) \prod_{\gamma=1}^{\alpha-1} a(v_\gamma, z_j) \prod_{k=1,\neq j}^{N} a^{-1}(z_k, z_j). \qquad \text{(IV.10)}
\end{aligned}
$$

Using the properties of determinant, one finds that if $j = i_2, \ldots, i_n$, the corresponding terms in (IV.10) contribute zero to the determinant. Thus, we may rewrite (IV.10) as

$$
\begin{aligned}
(\mathcal{B}^-_{\alpha 1})^{(1)} &= \sum_{j=1}^{N} \frac{e^{u_1 - v_\alpha} \sinh^2\eta}{\sinh(u_1 - z_j + \eta)\sinh(v_\alpha - z_j + \eta)} \prod_{\gamma=1}^{\alpha-1} \frac{\sinh(v_\gamma - z_j)}{\sinh(v_\gamma - z_j + \eta)} \\
&\quad \times \prod_{k=1,\neq j}^{N} \frac{\sinh(z_k - z_j + \eta)}{\sinh(z_k - z_j)} \\
&\equiv e^{u_1} f(u_1). \qquad \text{(IV.11)}
\end{aligned}
$$

Thanks to the Bethe ansatz equation (II.24), we may construct the function

$$\begin{aligned}
\mathcal{M}^{\pm}_{\alpha\beta} &= e^{\mp u_\beta} g(u_\beta) \\
&= \frac{e^{\pm(v_\alpha-u_\beta)}\sinh\eta}{\sinh(v_\alpha-u_\beta)} \prod_{\gamma=1}^{\alpha-1} \frac{\sinh(v_\gamma-u_\beta-\eta)}{\sinh(v_\gamma-u_\beta)} \left\{1-\prod_{k=1}^{N}\frac{\sinh(u_\beta-z_k)}{\sinh(u_\beta-z_k+\eta)}\right\}. \quad \text{(IV.12)}
\end{aligned}$$

Comparing $f(u_1)$ in (IV.11) with $g(u_1)$ in (IV.12), one finds that as functions of u_1, they have the same residues at the simple pole $u_1 = z_j - \eta \bmod(i\pi)$, and that when $u_1 \to \infty$, they are bounded. Moreover, one may prove that the residues of $g(u_1)$ at $u_1 = v_\nu$ $(\nu = 1, \ldots, \alpha)$ are zero because v_ν are solutions of the Bethe ansatz equation (II.24). Therefore, we have

$$(\mathcal{B}^{-}_{\alpha 1})^{(1)} = \mathcal{M}^{-}_{\alpha 1} = \frac{b^{-}(v_\alpha, u_1)}{a(v_\alpha, u_1)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(u_1, v_\gamma)\left(1-\prod_{k=1}^{N} a(u_1, z_k)\right). \quad \text{(IV.13)}$$

Then, by using the function $G^{(0)}, G^{(1)}$ and the intermediate function (IV.4) repeatedly, we obtain $G^{(m)}$ as

$$\begin{aligned}
&G^{(m)}(\{v_k\}, u_1, \ldots, u_m, i_{m+1}, \ldots, i_n) \\
&= (-1)^m (2\cosh\eta)^{\frac{n(n-1)-m(2n-m-1)}{2}} \prod_{1\le j<k\le m} a^{-1}(u_k, u_j) \\
&\quad \times \det(\mathcal{B}^{-})^{(m)}(\{v_k\}, u_1, \ldots, u_m, i_{m+1}, \ldots, i_n) \quad \text{(IV.14)}
\end{aligned}$$

with the matrix elements

$$\begin{aligned}
(\mathcal{B}^{-}_{\alpha\beta})^{(m)} &= \prod_{k=1}^{m} a(u_k, z_{i_\beta})\mathcal{B}^{-}_{\alpha\beta}, \qquad \text{for } \beta > m, \\
(\mathcal{B}^{-}_{\alpha\beta})^{(m)} &= \mathcal{M}^{-}_{\alpha\beta}, \qquad \text{for } \beta \le m. \quad \text{(IV.15)}
\end{aligned}$$

(IV.14) can be proved by induction. Firstly from (IV.8), (IV.9) and (IV.13), (IV.14) is true for $m = 1$. Assume (IV.14) for $G^{(m-1)}$. Let us show (IV.14) for general m. Substituting $G^{(m-1)}$ and (IV.5) into intermediate function (IV.4), we have

$$\begin{aligned}
&G^{(m)}(\{v_k\}, u_1, \ldots, u_m, i_{m+1}, \ldots, i_n) \\
&= \sum_{j\neq i_{m+1},\ldots,i_n}^{N} < i_n, \ldots, i_{m+1}|\tilde{B}(u_m)|i_{m+1}, \ldots, i_{m+p}, j, i_{m+p+1}, \ldots, i_n > \\
&\quad \times G^{(m-1)}(\{v_k\}, u_1, \ldots, u_{m-1}, i_{m+1}, \ldots, i_{m+p}, j, i_{m+p+1}, \ldots, i_n) \\
&= -(2\cosh\eta)^{-(n-m)} \sum_{j\neq i_{m+1},\ldots,i_n}^{N} b^{+}(u_m, z_j) \prod_{k\neq j}^{N} a^{-1}(z_k, z_j) \prod_{l=m+1}^{n} a(u_m, z_{i_l}) \\
&\quad \times G^{(m-1)}(\{v_k\}, u_1, \ldots, u_{m-1}, j, i_{m+1}, \ldots, i_n)
\end{aligned}$$

$$= (-1)^m (2\cosh\eta)^{\frac{n(n-1)-m(2n-m-1)}{2}} \prod_{1\le j<k\le m-1} a^{-1}(u_k, u_j)$$
$$\times \mathrm{det}\mathcal{B}'^{(m)}(\{v_k\}, u_1 \ldots, u_m, i_{m+1}, \ldots, i_n), \qquad \text{(IV.16)}$$

where the matrix elements $\mathcal{B}'^{(m)}_{\alpha\beta}$ are given by

$$\begin{aligned}
\mathcal{B}'^{(m)}_{\alpha\beta} &= \prod_{k=1}^{m} a(u_k, z_{i_\beta})\mathcal{B}^-_{\alpha\beta} \qquad \text{for } \beta > m, \\
\mathcal{B}'^{(m)}_{\alpha\beta} &= \mathcal{M}^-_{\alpha\beta} \qquad\qquad\qquad\quad \text{for } \beta < m, \\
\mathcal{B}'^{(m)}_{\alpha m} &= \prod_{i=1}^{m-1} a(u_i, z_j) \sum_{j\neq i_{m+1},\ldots,i_n} b^+(u_m, z_j) b^-(v_\alpha, z_j) \prod_{\gamma=1}^{\alpha-1} a(v_\gamma, z_j) \\
&\times \prod_{k=1,\neq j}^{N} a^{-1}(z_k, z_j). \qquad \text{(IV.17)}
\end{aligned}$$

Thus, by the procedure leading to $(\mathcal{B}^-_{\alpha\beta})^{(1)}$, we can prove

$$\mathcal{B}'^{(m)}_{\alpha m} = \prod_{i=1}^{m-1} a^{-1}(u_m, u_i)\mathcal{M}^-_{\alpha m}. \qquad \text{(IV.18)}$$

Then one sees that $\mathcal{B}'^{(m)}_{\alpha\beta} = \mathcal{B}^{(m)}_{\alpha\beta}$. Therefore we have proved that (IV.14) holds for all m.

When $m = n$, we obtain the scalar product $S_n(\{u_i\}, \{v_j\})$,

$$S_n(\{u_i\}, \{v_j\}) = (-1)^n \prod_{k>l}^{n} a^{-1}(u_k, u_l)\mathrm{det}\mathcal{M}^-(\{v_j\}, \{u_i\}), \qquad \text{(IV.19)}$$

where the matrix elements of $\mathcal{M}^-$ are given by

$$\mathcal{M}^\pm_{\alpha\beta} = \frac{b^\pm(v_\alpha, u_\beta)}{a(v_\alpha, u_\beta)} \prod_{\gamma=1}^{\alpha-1} a^{-1}(u_\beta, v_\gamma) \left(1 - \prod_{k=1}^{N} a(u_\beta, z_k)\right). \qquad \text{(IV.20)}$$

By using the expression of the eigenvalues of the system (II.23), the scalar product (IV.19) can be rewritten as

$$S_n(\{u_i\}, \{v_j\}) = (-1)^n \prod_{k>l}^{n} a^{-1}(u_k, u_l)\mathrm{det}\hat{\mathcal{M}}^-(\{v_j\}, \{u_i\}) \qquad \text{(IV.21)}$$

with the matrix $\hat{\mathcal{M}}^\pm$ being

$$\hat{\mathcal{M}}^\pm_{\alpha\beta} = e^{\pm(v_\alpha - u_\beta)} \sinh(u_\beta - v_\alpha) \prod_{\mu\neq\alpha} a(v_\mu, u_\beta) \prod_{\gamma=1}^{\alpha-1} a^{-1}(u_\alpha, v_\gamma) \frac{\partial\Lambda(u_\beta, \{v_\alpha\})}{\partial v_\alpha}. \qquad \text{(IV.22)}$$

Remark: In the derivation of (IV.19), the parameters v_i in the state $\tilde{C}(v_1)\ldots\tilde{C}(v_n)|0>$ are required to satisfy the BAE (II.24). However, the parameters u_j $(j = 1, \ldots, n)$ in the dual state $< 0|\tilde{B}(u_n)\ldots\tilde{B}(u_1)$ do not need to satisfy the BAE.

On the other hand, if we compute the scalar product by starting from the dual state $< 0|B(v_n)\ldots B(v_1)$, then by using the same procedure, we have

$$\begin{aligned} S_n(\{v_i\},\{u_j\}) &= \ < 0|\tilde{B}(v_n)\ldots\tilde{B}(v_1)\tilde{C}(u_1)\ldots\tilde{C}(u_n)|0> \\ &= \ (-1)^n\prod_{k>l}^{n}a^{-1}(u_k,u_l)\,\mathrm{det}\mathcal{M}^+(\{v_i\},\{u_j\}). \end{aligned} \tag{IV.23}$$

In the above equation, we have assumed that $\{v_i\}$ satisfy the BAE.

Noticing the BAE (II.24), one sees that the scalar product $S_n(\{u_i\},\{v_j\}) = 0$ if both parameter sets $\{u_i\}$ and $\{v_j\}$ ($\{v_j\} \neq \{u_i\}$ $i,j = 1,\ldots,n$) in (IV.19) and (IV.23) satisfy the BAE.

Let $u_\alpha \to v_\alpha$ ($\alpha = 1,\ldots,n$) in (IV.19), we obtain the Gaudin formula for the norm of the $U_q(gl(1|1))$ Bethe state.

$$\begin{aligned} \mathcal{S}_n &= S_n(\{v_j\},\{v_k\}) =< 0|B(v_n)\ldots B(v_1)C(v_1)\ldots C(v_n)|0> \\ &= (-1)^n\sinh^n\eta\prod_{k>j}^{n}\frac{\sinh^2(v_k-v_j+\eta)}{\sinh^2(v_k-v_j)}\left[\prod_{\alpha=1}^{n}\frac{1}{v_\alpha-u_\alpha}\left(1-\prod_{l=1}^{N}\frac{\sinh(u_\alpha-z_l)}{\sinh(u_\alpha-z_l+\eta)}\right)\right]_{u_\alpha\to v_\alpha} \\ &= (-1)^n\sinh^n\eta\prod_{k>j}^{n}\frac{\sinh^2(v_k-v_j+\eta)}{\sinh^2(v_k-v_j)}\left[\prod_{\alpha=1}^{n}\frac{\partial}{\partial u_\alpha}\ln\left(\prod_{l=1}^{N}\frac{\sinh(u_\alpha-z_l)}{\sinh(u_\alpha-z_l+\eta)}\right)\right]_{u_\alpha\to v_\alpha} \\ &= (-1)^n\sinh^{2n}\eta\prod_{k>j}^{n}\frac{\sinh^2(v_k-v_j+\eta)}{\sinh^2(v_k-v_j)}\prod_{\alpha=1}^{n}\sum_{l=1}^{N}\frac{1}{\sinh(v_\alpha-z_l)\sinh(v_\alpha-z_l+\eta)}, \end{aligned} \tag{IV.24}$$

where we have used the BAE (II.24).

V Correlation functions

Having obtained the scalar product and the norm, we are now in the position to compute the k-point correlation functions of the model. In general, a k-point correlation function is defined by

$$F_n^{\epsilon^1,\ldots,\epsilon^k} =< 0|B(u_n)\ldots B(u_1)\epsilon^1_{i_1}\ldots\epsilon^k_{i_k}C(v_1)\ldots C(v_n)|0>, \tag{V.1}$$

where $\epsilon^j_{i_j}$ stand for the local fermion operators $c_{i_j}, c^\dagger_{i_j}$ or n_{i_j}, and the lower indices i_j indicate the positions of the fermion operators.

The authors in [16] proved that the local spin and field operators of the fundamental graded models can be represented in terms of monodromy matrix. Specializing to the current system, we obtain

$$c_j^\dagger = \prod_{k=1}^{j-1}(-A(z_k)+D(z_k))\cdot B(z_j)\cdot \prod_{k=j+1}^{N}(-A(z_k)+D(z_k)), \quad \text{(V.2)}$$

$$c_j = \prod_{k=1}^{j-1}(-A(z_k)+D(z_k))\cdot C(z_j)\cdot \prod_{k=j+1}^{N}(-A(z_k)+D(z_k)), \quad \text{(V.3)}$$

$$n_j = \prod_{k=1}^{j-1}(-A(z_k)+D(z_k))\cdot D(z_j)\cdot \prod_{k=j+1}^{N}(-A(z_k)+D(z_k)). \quad \text{(V.4)}$$

V.1 One point functions

In this subsection, we compute the one point functions for the local operators $c_m^\dagger$, c_m and n_m, respectively.

We first calculate $c_m^\dagger$. Noticing that the Bethe state and its dual are eigenstates of the transfer matrix under the constraint of the BAE, we have, from (V.2),

$$\begin{aligned}
& F_n^-(\{u_j\},z_m,\{v_k\}) \\
=\; & <0|B(u_n)\ldots B(u_1)c_m^\dagger C(v_1)\ldots C(v_{n+1})|0> \\
=\; & \phi_{m-1}(\{u_j\})\phi_m^{-1}(\{v_k\}) <0|B(u_n)\ldots B(u_1)B(z_m)C(v_1)\ldots C(v_{n+1})|0> \\
=\; & \phi_{m-1}(\{u_j\})\phi_m^{-1}(\{v_k\}) <0|\tilde{B}(u_n)\ldots \tilde{B}(u_1)\tilde{B}(z_m)\tilde{C}(v_1)\ldots \tilde{C}(v_{n+1})|0> \\
=\; & \phi_{m-1}(\{u_j\})\phi_m^{-1}(\{v_k\})S_{n+1}(u_n,\ldots,u_1,z_m,\{v_j\}) \\
=\; & (-1)^{n+1}\phi_{m-1}(\{u_j\})\phi_m^{-1}(\{v_k\})\prod_{k>j}^{n}a^{-1}(u_k,u_j)\prod_{l=1}^{n}a^{-1}(u_l,z_m) \\
& \times \det\mathcal{M}^-(\{v_j\},z_m,u_1,\ldots,v_n),
\end{aligned} \quad \text{(V.5)}$$

where $\phi_i(\{u_j\}) = \prod_{k=1}^{i}\prod_{l=1}^{n}a(u_l,u_k)$. As mentioned in the remark of the previous section, $F_n^- = 0$ if the parameter set $\{u_i\}$ $(i=1,\ldots,n)$ is not a subset of $\{v_j\}$ $(j=1,\ldots,n+1)$. When $\{u_i\}\subset\{v_j\}$, (V.5) can be simplified to a simple function. For example, if $u_i = v_{i+1}$ $(i=1,\ldots,n)$, the one point function F^- becomes

$$\begin{aligned}
& F_n^-(v_{n+1},\ldots,v_2,z_m,v_1,\ldots,v_{n+1}) \\
=\; & (-1)^{n+1}\frac{\phi_{m-1}(\{u_j\})}{\phi_m(\{v_k\})}\frac{e^{-(v_1-z_m)}\sinh^{2n+1}\eta}{\sinh(v_1-z_m)}\prod_{k>j=2}^{n+1}\frac{\sinh^2(v_k-v_j+\eta)}{\sinh^2(v_k-v_j)}\prod_{j=2}^{n+1}\frac{\sinh(v_j-z_m+\eta)}{\sinh(v_j-z_m)} \\
& \times\prod_{j=2}^{n+1}\frac{\sinh(v_j-v_1+\eta)}{\sinh(v_j-v_1)}\prod_{\alpha=2}^{n+1}\sum_{l=1}^{N}\frac{1}{\sinh(v_\alpha-z_l)\sinh(v_\alpha-z_l+\eta)}.
\end{aligned} \quad \text{(V.6)}$$

Similarly, when $\{u_i\} \subset \{v_j\}$, we obtain the one-point function involving the operator c_m:

$$\begin{aligned}
& F_n^+(\{v_k\}, z_m, \{u_j\}) \\
= \; & < 0|B(v_{n+1}) \dots B(v_1) c_m C(u_1) \dots C(u_n)|0 > \\
= \; & \phi_{m-1}(\{v_j\}) \phi_m^{-1}(\{u_k\}) S_{n+1}(\{v_j\}, z_m, u_1, \dots, u_n) \\
= \; & (-1)^{n+1} \phi_{m-1}(\{v_j\}) \phi_m^{-1}(\{u_k\}) \prod_{k>j}^{n} a^{-1}(u_k, u_j) \prod_{l=1}^{n} a^{-1}(u_l, z_m) \\
& \times \det \mathcal{M}^+(\{v_j\}, z_m, u_1, \dots, v_n).
\end{aligned} \tag{V.7}$$

F_n^+ is non-vanishing if $\{u_i\} \subset \{v_j\}$. When $\{u_i\} \subset \{v_j\}$, (V.7) can also be simplified to a simple function. In the case $u_i = v_{i+1}$ $(i = 1, \dots, n)$, the one point function F^+ becomes

$$\begin{aligned}
& F_n^+(v_{n+1}, \dots, v_2, z_m, v_1, \dots, v_{n+1}) \\
= \; & (-1)^{n+1} \frac{\phi_{m-1}(\{v_j\})}{\phi_m(\{u_k\})} \frac{e^{(v_1 - z_m)} \sinh^{2n+1} \eta}{\sinh(v_1 - z_m)} \prod_{k>j=2}^{n+1} \frac{\sinh^2(v_k - v_j + \eta)}{\sinh^2(v_k - v_j)} \prod_{j=2}^{n+1} \frac{\sinh(v_j - z_m + \eta)}{\sinh(v_j - z_m)} \\
& \times \prod_{j=2}^{n+1} \frac{\sinh(v_j - v_1 + \eta)}{\sinh(v_j - v_1)} \prod_{\alpha=2}^{n+1} \sum_{l=1}^{N} \frac{1}{\sinh(v_\alpha - z_l) \sinh(v_\alpha - z_l + \eta)}.
\end{aligned} \tag{V.8}$$

The one-point function involving the operator n_m is defined by

$$F_n^{n_m}(\{u_j\}, z_m, \{v_k\}) =< 0|B(u_n) \dots B(u_1) n_m C(v_1) \dots C(v_{n+1})|0 > . \tag{V.9}$$

Substituting (V.4) into the above equation and considering the BAE, we have

$$\begin{aligned}
& F_n^{n_m}(\{u_j\}, z_m, \{v_k\}) =< 0|B(u_n) \dots B(u_1) n_m C(v_1) \dots C(v_n)|0 > \\
= \; & \frac{\phi_{m-1}(\{u_j\})}{\phi_{m-1}(\{v_k\})} < 0|\tilde{B}(u_n) \dots \tilde{B}(u_1) \tilde{D}(z_m) \tilde{C}(v_1) \dots \tilde{C}(v_n)|0 > .
\end{aligned} \tag{V.10}$$

With the help of (II.21), we see

$$\begin{aligned}
& D(z_m) C(v_1) \dots C(v_n)|0 > \\
= \; & \prod_{k=1}^{n} a^{-1}(v_k, z_m) C(v_1) \dots C(v_n)|0 > \\
& - \sum_{j=1}^{n} \frac{b^-(v_j, z_m)}{a(v_j, z_m)} \prod_{l=1}^{j-1} \frac{c(v_l, v_j)}{c(v_l, z_m)} \prod_{k=1, \neq j}^{n} a^{-1}(v_k, v_j) \\
& \times C(v_1) \dots C(v_{j-1}) C(z_m) C(v_{j+1}) \dots C(v_n)|0 > .
\end{aligned} \tag{V.11}$$

Therefore, substituting (V.11) into (V.10), we obtain

$$
\begin{aligned}
& F_n^{n_m}(\{u_j\}, z_m, \{v_k\}) \\
= \; & \frac{\phi_{m-1}(\{u_j\})}{\phi_{m-1}(\{v_k\})} \prod_{k=1}^{n} a^{-1}(v_k, z_m) S_n(\{u_i\}, \{v_j\}) \\
& - \sum_{j=1}^{n} \frac{b^-(v_j, z_m)}{a(v_j, z_m)} \prod_{l=1}^{j-1} \frac{c(v_l, v_j)}{c(v_l, z_m)} \prod_{k=1, \neq j}^{n} a^{-1}(v_k, v_j) S_n(\{u_i\}, v_1, \ldots, v_{j-1}, z_m, v_{j+1}, \ldots, v_n) \\
= \; & (-1)^n \frac{\phi_{m-1}(\{u_j\})}{\phi_{m-1}(\{v_k\})} \prod_{k=1}^{n} a^{-1}(v_k, z_m) \prod_{k>j} a^{-1}(v_k, v_j) \\
& \times \det\left[M^+(\{u_i\}, \{v_j\}) - \mathcal{N}(\{u_i\}, \{v_j\}, z_m) \right], \qquad \text{(V.12)}
\end{aligned}
$$

where $\mathcal{N}$ is a rank-one matrix with the following matrix elements

$$
\mathcal{N}_{\alpha\beta}(\{u_i\}, \{v_j\}, z_m) \;=\; \frac{e^{u_\alpha - v_\beta} \sinh^2 \eta}{\sinh(u_\alpha - z_m) \sinh(v_\beta - z_m + \eta)} \prod_{i=1}^{\alpha-1} \frac{\sinh(z_m - u_i + \eta)}{\sinh(z_m - u_i)}. \qquad \text{(V.13)}
$$

In the above derivation, we have used the following property of determinant: If $\mathcal{A}$ is an arbitrary $n \times n$ matrix and $\mathcal{B}$ is a rank-one $n \times n$ matrix, then the determinant of $\mathcal{A} + \mathcal{B}$ is given by

$$
\det(\mathcal{A} + \mathcal{B}) = \det\mathcal{A} + \sum_{i=1}^{n} \det\mathcal{A}^{(i)}, \qquad \text{(V.14)}
$$

where

$$
\begin{aligned}
\mathcal{A}^{(i)}_{\alpha\beta} &= \mathcal{A}_{\alpha\beta} \qquad \text{for } \beta \neq i, \\
\mathcal{A}^{(i)}_{\alpha i} &= \mathcal{B}_{\alpha i}.
\end{aligned}
$$

V.2 Correlation function of two adjacent operators

In the subsection, we compute the correlation function of two adjacent operators c_m and c_{m+1} defined by

$$
F_n^{-+}(\{u_i\}, z_m, z_{m+1}, \{v_j\}) = < 0 | B(u_n) \ldots B(u_1) c_m c^\dagger_{m+1} C(v_1) \ldots C(v_n) | 0 > \;. \qquad \text{(V.15)}
$$

Substituting (V.3) and (V.2) into the above definition and considering the fact $\prod_{k=1}^{N} t(z_k) = 1$, we have

$$
\begin{aligned}
& F_n^{-+}(\{u_i\}, \{v_j\}, z_m, z_{m+1}) \\
& = \frac{\phi_{m-1}(\{u_i\})}{\phi_{m+1}(\{v_j\})} < 0 | \tilde{B}(u_n) \ldots \tilde{B}(u_1) \tilde{C}(z_m) \tilde{B}(z_{m+1}) \tilde{C}(v_1) \ldots \tilde{C}(v_n) | 0 > \;. \qquad \text{(V.16)}
\end{aligned}
$$

By using the commutation relation (II.22), we obtain

$$
\begin{aligned}
&B(z_{m+1})C(v_1)\ldots C(v_n)|0>= (-1)^n C(v_1)\ldots C(v_n)B(z_{m+1})|0> \\
&+\sum_{j=1}^{n}(-1)^{j+1}\frac{b^+(z_{m+1},v_j)}{a(z_{m+1},v_j)}C(v_1)\ldots C(v_{j-1})D(z_{m+1})t(v_j)C(v_{j+1})C(v_n)|0> \\
&+\sum_{j=1}^{n}(-1)^{j}\frac{b^+(z_{m+1},v_j)}{a(z_{m+1},v_j)}C(v_1)\ldots C(v_{j-1})t(z_{m+1})D(v_j)C(v_{j+1})C(v_n)|0>, \qquad \text{(V.17)}
\end{aligned}
$$

where $\tilde{t}(u) \equiv F_{1\ldots N}t(u)F^{-1}_{1\ldots N}$. On the rhs of the above equation, one easily finds that the first term is zero. Using the BAE, one may check that the second term also equals to zero. Therefore, only the third term survives on the rhs of the above equation and we have

$$
\begin{aligned}
&B(z_{m+1})C(v_1)\ldots C(v_n)|0> \\
&=\sum_{j=1}^{n}(-1)^{j}\,\frac{b^+(z_{m+1},v_j)}{a(z_{m+1},v_j)}\prod_{k=j+1}^{n}a^{-1}(v_k,z_{m+1})\prod_{l=j+1}^{n}a^{-1}(v_l,v_j) \\
&\quad\times C(v_1)\ldots C(v_{j-1})C(v_{j+1})C(v_n)|0> \\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}\,\frac{b^+(z_{m+1},v_j)}{a(z_{m+1},v_j)}\prod_{k=j+1}^{n}a^{-1}(v_k,z_{m+1}) \\
&\quad\times\sum_{l=j+1}^{n}\frac{b^-(v_l,v_j)}{a(v_l,v_j)}\prod_{m=j+1}^{l-1}\frac{c(v_m,v_l)}{c(v_m,v_j)}\prod_{i=j+1,\neq l}^{n}a^{-1}(v_i,v_l) \\
&\quad\times C(v_1)\ldots C(v_{j-1})C(v_{j+1})\ldots C(v_{l-1})C(v_j)C(v_{l+1})\ldots C(v_n)|0> \\
&\equiv\sum_{j=1}^{n}M_jC(v_1)\ldots C(v_{j-1})C(v_{j+1})C(v_n)|0> \\
&\quad+\sum_{j=1}^{n}\sum_{l=j+1}^{n}M_{j,l}C(v_1)\ldots C(v_{j-1})C(v_{j+1})\ldots C(v_{l-1})C(v_j)C(v_{l+1})\ldots C(v_n)|0>. \\
&\hfill \text{(V.18)}
\end{aligned}
$$

Substituting (V.18) into (V.16), we obtain two-point correlation function F_n^{-+}

$$
\begin{aligned}
&F_n^{-+}(\{u_i\},z_m,z_{m+1},\{v_j\}) \\
&=\frac{\phi_{m-1}(\{u_i\})}{\phi_{m+1}(\{v_j\})}\left[\sum_{j=1}^{n}M_jS_n(\{u_i\},z_m,v_1,\ldots,v_{j-1},v_{j+1},v_n)\right. \\
&\quad\left.+\sum_{j=1}^{n}\sum_{l=j+1}^{n}M_{j,l}S_n(\{u_i\},z_m,v_1,\ldots,v_{j-1},v_{j+1},\ldots,v_{l-1},v_j,v_{l+1},\ldots,v_n)\right]. \qquad \text{(V.19)}
\end{aligned}
$$

VI Discussion

In this paper, with the help of the factorizing F-matrix (F-basis), we have obtained the determinant representations of the scalar products and correlation functions of the $U_q(gl(1|1))$ free fermion model.

In [10]-[15], the authors studied the correlation functions of the free fermion model based on the finite XX0 spin chain (XY model [9]) with periodic boundary condition

$$H_{XX0} = \sum_{j=1}^{N} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h\sigma_j^z \right), \tag{VI.1}$$

where σ^ϵ ($\epsilon = x, y, z$) are the Pauli matrices and h is an external classical magnetic field. The equivalence between the free fermion model and the XX0 model can be proved by using the Jordan-Wigner transform

$$c_k = \exp[i\pi Q_{k-1}]\sigma_k^+, \tag{VI.2}$$

$$c_k^\dagger = \sigma_k^- \exp[i\pi Q_{k-1}], \tag{VI.3}$$

where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \sigma^y)$, $Q_k = \sum_{j=1}^{k} \frac{1}{2}(1 - \sigma_k^z)$. Because of the periodic boundary condition of the finite XX0 spin chain, we have

$$\sigma_{N+1}^\pm = \sigma_1^\pm. \tag{VI.4}$$

Substituting the Jordan-Wigner transforms into the above relation, we obtain

$$c_{N+1} = \exp[i\pi Q_N]c_1, \quad c_{N+1}^\dagger = c_1^\dagger \exp[i\pi Q_N]. \tag{VI.5}$$

Thus, comparing the above boundary condition with that of the $U_q(gl(1|1))$ free fermion model (II.14), we find that the free fermion model arising from the XX0 model has a twisted boundary condition which depends on the operator $\sigma^z = \sum_{i=1}^{N} \sigma_i^z$.

On the other hand, by means of the Jordan-Wigner transform, the $U_q(gl(1|1))$ free fermion model is equivalent to a twisted XX0 model, and the one-point correlation functions (V.5) and (V.7) give rise to the m-point correlation functions of the twisted XX0 model. For example: substituting (VI.3) into (V.5), we obtain

$$\begin{aligned} & F_n^-(\{u_j\}, z_m, \{v_k\}) \\ = & < 0|B(u_n) \ldots B(u_1) c_m^\dagger C(v_1) \ldots C(v_{n+1})|0 > \\ = & < 0|B(u_n) \ldots B(u_1) \sigma_1^z \ldots \sigma_{m-1}^z \sigma_m^- C(v_1) \ldots C(v_{n+1})|0 > . \end{aligned} \tag{VI.6}$$

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